

Valuation of asset and volatility-dependent derivatives using decoupled time-changed Lévy processes ^{*†}

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Abstract

A decoupled time-changed (DTC) Lévy process is a generalized time-changed Lévy process whose continuous and discontinuous parts are allowed to follow separate random time scalings. Disentangling the stochastic time change in two components not only provides a powerful unitary framework for the models already present in the literature, but in principle opens up new possibilities for the choice of an asset's log-price dynamics. Following this new idea, we devise a general martingale structure for the price process, and obtain an inverse-Fourier pricing equation for claims paying at maturity a function of the asset value *and* its realized volatility. Thus with a single formula we are able to capture prices for derivatives depending on either an asset or its volatility, but also joint payoffs are allowed, like the *target volatility option* (TVO), which shall be discussed. Numerical computations validating our techniques are provided. Notably, the DTC theory allows to incorporate valuation formulae from various asset models in a single software implementation.

Keywords: Derivative pricing; time changes; Lévy processes; volatility derivatives; target volatility option; martingale representation theory.

AMS subject classifications: 91G20, 60G46, 60G40.

1 Introduction

The aim of this paper is to illustrate and solve, under a class of generalized Lévy models, the pricing problem for a contingent claim jointly depending on a market asset S_t and its realized volatility RV_t . The contribution of this work seems to us as being two-fold:

- (i). It defines a class of asset price models that naturally extends the existing theory of time-changed Lévy models;
- (ii). It introduces the class of derivatives paying off on an asset *and* the volatility in mathematically rigorous terms, and provides the necessary machinery for pricing contingent claims of such kind in the presented stochastic framework.

Although these two ideas do represent separate research topics, they will be developed in this article through a common discussion. In particular, the elaboration of the first point will be functional to the fulfillment of the second.

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1.1 Time changes

The use of Lévy models in finance dates back to 1976 when Merton, in [36], proposed that the log-price dynamics of a stock return should follow an exponential Brownian diffusion punctuated by a Poisson arrival process of normally distributed jumps. For the first time then the two main shortcomings of the Black-Scholes model, continuity of the sample paths and normality of returns, were addressed. Ever since, Lévy processes have proved to be a flexible and yet mathematically tractable instrument for asset price sampling and modeling. To name just a few, popular models are the Variance Gamma (Madan *et al.*, [34]), the CGMY (Carr *et al.*, [7]), and the Normal Inverse Gaussian (Barndorff-Nielsen, [4]).

In finance, the ideas of Lévy processes and *time changes* were linked from the very beginning. Indeed, one of the easiest way of producing Lévy processes is using the principle of *subordination* of a Brownian motion W_t . If T_t is an increasing Lévy process independent of W_t , then the subordinated process W_{T_t} will still be of Lévy type; most importantly, its characteristic function is easily recovered from those of T_t and X_t . Ultimately, this means that asset models based on exponentials of subordinated Lévy processes can be obtained from the Lévy structures of W_t and T_t with little effort.

Subordination is the simplest example of a time change. Given a process X_t and an increasing process T_t , the time change of X_t according to T_t is the process X_{T_t} . Stock price return models depending on time-changed Brownian motions have been conjectured for the first time by Clark in [9]. Further theoretical evidence supporting the financial use of time-changed models is given by Monroe's celebrated Theorem in [37], asserting that any semimartingale can be viewed as a time change of a Brownian motion. As a consequence, any semimartingale representing the log-price process of an asset can be considered as a re-scaled Wiener process. Empirical studies (Ané and Geman, [1]) confirmed that normality of returns can be recovered in a new price density based on the quantity and arrival times of orders, which justifies the heuristic interpretation of T_t as “*business time*” or “*stochastic clock*”.

Further advances were later made by Carr and Wu in [8]. They demonstrated that much more general time changes are potential candidates asset price modeling. Consequently, they re-interpreted a tremendous amount of models from the literature as time changes. Their theory effectively encompasses many of the areas of asset modeling research: finite/infinite activity Lévy models, stochastic volatility models, and volatility/asset correlations effects (the so-called *leverage effect*). The latter can be captured in the valuation process by introducing a complex measure change inducing distributions of a market with no leverage; such a probability measure is known as the *leverage-neutral measure*.

However, Carr and Wu's work still leaves out a number of popular possibilities. For example, the widely known model by Bates ([2], [3]) and the relatively more sophisticated one by Fang, [17], are not time-changed Lévy as understood in [8]. Indeed, in these two models the jump component does not follow the same time scaling as the continuous Brownian part. In the Bates model the discontinuities have stationary increments, whereas in the Fang model the jump rate is allowed to follow a diffusion of his own. In other words, price models for which the “stochastic clock” runs at different paces for the “small” and “big” market movements have already been proposed and tested. Statistical analyses (Bates [3], Fang [17], Sec. 4) confirm that these models are capable of improved data fitting.

Motivated by such instances, the natural question arising is then whether it is possible to manufacture consistent general time-changed price processes in which the continuous and discontinuous parts of the underlying Lévy model follow two *different* stochastic time changes.

We shall show that the answer is affirmative. Effectively, we extend the idea of [8] of encapsulating different classes of models in a single time-changed representation to an even broader range of stochastic processes, and show that these are suitable for financial uses. Moreover, we are still able to capture the correlation structures between the “stochastic clocks” and the underlying Lévy process by a measure change.

The class of models we propose is obtained by time-modifying the continuous and jump parts of a given Lévy process X_t by two (not necessarily independent) stochastic time scalings T_t and U_t satisfying a certain regularity condition (Definition 3.1). We call this a *decoupled time change*. In a formula:

$$X_{T,U} := X_{T_t}^c + X_{U_t}^d, \quad (1.1)$$

where X_t^c and X_t^d represent respectively the Brownian and jump components of X_t . Since any time-changed Lévy process X_{T_t} is in particular a decoupled time change, this clearly provides a natural generalization of the theory of [8]. How to derive martingale relations for a stock whose log-returns are modeled by $X_{T,U}$ will be explained in section 2.

1.2 Joint asset/volatility derivatives

Volatility derivatives took off in the late nineties mostly as a market answer to the post-LTCM crisis. Back then hedge funds started selling variance swaps at the high market implied volatility levels, and other counterparties were happy to buy these products in order to protect their portfolios against possible implied volatility increases. The realized volatility of an asset over a period of n days is empirically computed as the standard deviation of the daily log-returns:

$$RV_n = \sqrt{\frac{1}{n} \sum_{t_i=1}^n |\log S_{t_{i+1}} - \log S_{t_i}|^2}. \quad (1.2)$$

For example, trading a volatility swap maturing in n days necessitates the computation of the fair price for delivery of RV_n , times a notional amount, in n days. If S_t is an equity, the short side of an investor's portfolio involving S_t would be hurt by an increase of the terms in the sum (1.2) (vega effect), but going long a volatility swap on S_t may offset the corresponding loss. Compared to delta-hedging the position, this strategy has the obvious advantage of eliminating the frictions and dangers typically associated with portfolio rebalancing.

A vast literature on volatility derivatives pricing consequently developed. To name a few, Dupire in [14] was the first to note that a contract on the realized variance can be replicated in a model-independent way. Neuberger [38] and Demeterfi *et al.* [11] proposed the replication techniques of a variance swap via the log-contract, an idea which was widely expanded in the pricing by robust replication paper [6] by Carr and Lee. For methods relying on specific models see for instance Matytsin [35], Elliott *et al.* [16], Javaheri [25] *et al.*

A more recent (as of 2008 to our knowledge) market innovation, is that of derivatives and investment strategies based on volatility-modified versions of plain vanilla products. Such contracts are able to replicate classic European payoffs under a perfect volatility foresight. At the same time, the component of the price due to a vega excess may be reduced by using the realized volatility as a normalizing factor. One example of such a product is the *target volatility option*. A target volatility (call) option (TVO) pays at maturity t the value:

$$F(S_t, RV_t) = \frac{\bar{\sigma}}{\sqrt{RV_t}} (S_t - K)^+, \quad (1.3)$$

for a strike price K and some other constant $\bar{\sigma}$ written in the contract. Intuitively, the closer RV_t will be to $\bar{\sigma}$, the more this product will behave like a call option. However, the presence of RV_t at the denominator depresses the sensitivity of F to a change in volatility. To give just an instance of how this could be beneficial, in times of market hardship a TVO allows an investor to take a long position in an option format without having to incur in the high costs caused by implied volatility increases. In section 7 we will investigate more the target volatility option and use it as benchmark product for numerical validation of our formulae.

Motivated by the presence of this new class of derivative securities, it is this authors' interest to pursue a comprehensive study, already begun in [42], of general joint asset/volatility contingent claims. Instruments of this kind can be represented, in terms of their underlying stochastic components, in the form $F(S_t, RV_t)$ for a measurable function F of two real variables.

The problem of pricing joint volatility/asset derivatives in general, and the target volatility option in particular, has been already addressed by Di Graziano and Torricelli in [12] for a 0-correlation stochastic volatility model, and by the author in [42], for a general stochastic volatility model. We will recover some of the results from the latter in the present work.

1.3 Pricing

As already stressed, the decoupled time change approach, although having theoretical relevance of its own, has been here conceived as *pricing-oriented*. Traditionally, mathematical finance presents three main research strains for securities pricing: numerical simulation, PDE solution approximation methods, and implementation of analytical formulae. Regarding the latter, closed versions are rarely available, the Black-Scholes formula being one notable exception. However, because of the nature of the distributions for the underlying price densities, analytical representations are much more likely to exist for the *Fourier transform* of the solution (e.g. Heston model, Lévy processes). Having

such formulae at hand, the valuation problem is simply solved by numerically performing the integration inverting the transform.

Fourier methods for contingent claims valuation have been appearing in the literature since the pioneering work of Heston, [22], who was the first to reckon that the inversion of a characteristic solution of a PDE naturally led to Option prices.

Other important Fourier-inversion methods are given in the FFT paper [5] by Carr and Madan, and in Lewis' book [32] and successive paper [33]. The advantage of Lewis' approach is that it is valid for all payoffs whose transform is known in closed formula, and that it most clearly differentiates between the *payoff* and *model* pricing components. Furthermore, it provides an easy way for calculating price sensitivities.

Lewis' original idea of [32] is in fact a no-nonsense approach to claim valuation. Solving a pricing problem is equivalent to finding a PDE solution, which is in turn given by the integral of a fundamental solution times the terminal condition. In an equivalent dual formulation, the solution can be found by the inversion integral of the Fourier transform of the fundamental solution (the "*fundamental transform*", a characteristic function from the process) multiplied by the terminal condition's (payoff) Fourier transform. An analogous argument can be used (see [33]) when the underlying dynamics do not arise from a diffusion problem but are instead directly derived from a Lévy model.

As we will show, it turns out that these ideas have an *ad verbatim* multivariate extension whenever the payoff and the characteristic function from the model include dependence on the quadratic variation of the log-price process. This will be the technique we will favour here for contingent claims pricing.

Obviously, our proofs are in particular valid for plain contingent claims on an asset S_t ; nonetheless, introducing a volatility variable in the payoff can be dealt with by a minimal effort. It is also clear that the results we derive equally apply for pricing of classic volatility derivatives, like volatility and variance swaps. Indeed, by reverting to the case $T_t = U_t$ and considering payoff functions in which the first argument does not appear, it is possible to analytically price pure volatility derivatives in the standard time-changed Lévy processes theory.

1.4 Theoretical perspective and structure of the paper

The technical approach to time-changing featured in this paper is rather different from that found in the previous research. In our opinion, some of the existing literature does not take into sufficient account the complications that may arise in time-changed asset modeling. In a nutshell, these owe to the fact that stochastic time transformations are notoriously ill-behaved with respect to the martingale property. Lacking this, it is impossible to even talk about risk-neutral valuation. Note that in this regard the fact that X_t is a Lévy process is irrelevant. For example, a Brownian motion re-scaled with the time change given by the first hitting times of the levels $t > 0$ (an inverse Gaussian process) becomes the deterministic process t itself. The key requirement to preserve the martingale property after a time change is the *uniform integrability* of the underlying process X_t : a uniformly integrable martingale remains a martingale after a time change; this is due to Doob's optional sampling Theorem. When such an assumption is dropped, counterexamples are known (see the discussion after Proposition 3.3).

Unfortunately exponential martingales, the fundamental structures of asset modeling, do not normally enjoy the uniform integrability property. For instance a geometric Brownian motion, historically the most prominent asset price model, is not uniformly integrable on \mathbb{R}_+ .

Of course, lack of uniform integrability is ultimately not so important in a model where time runs linearly. Indeed, for financial purposes we always look at an asset evolution in a finite time period. Still, whenever X_t must be subject to a time change, we cannot afford to enforce this limitation, otherwise for some finite values of t the set $\{T_t = +\infty\}$ may have positive probability. That is, T_t is no longer a valid time change when restricting X_t to a bounded time interval.

A shift of paradigm seems advisable. Throughout this paper we hope to provide convincing evidence that (decoupled) time-changed processes are much more conveniently looked at in the bigger picture of the semimartingale representation theory.

The change of approach we propose effectively avoids any loose arguments when time-changing non-uniformly integrable martingales. Insisting on an exponential martingale to remain a martingale after a time change is tempting,

but simply not possible, because Doob's Theorem may not apply. Instead, we shall identify a one-parameter family of exponential *local* martingales canonically associated with X_{T_t} , whose elements turn out to be true martingales on a certain subset Θ_0 of admissible parameters. Determining the shape of Θ_0 is in general a difficult problem. Sufficient conditions are in some cases given; examples are the famous *Novikov* and *Kazamaki* conditions for exponentials of Brownian integrals, which we will recover as particular cases.

The matter of this article is organized as follows. In section 2 we lay out the assumptions that we shall need; in section 3 martingale properties for a decoupled time-changed Lévy model will be derived. Section 4 shows the fundamental relation between the characteristic function of the joint transition density of $(S_t, \langle \log S \rangle_t)$ and the joint Laplace transform of (T_t, U_t) as computed in an appropriate measure. Section 5 is dedicated to the derivation of a pricing formula for products paying off on S_t and $\langle \log S \rangle_t$. We devote section 6 to explaining some of the models present the literature in terms of a decoupled time change, and find the joint characteristic function of $(S_t, \langle \log S \rangle_t)$ for each such model. Finally, in section 7 we use our formulae to value a TVO under various market conditions and asset models, and briefly summarize the work done. The proofs are found in the appendix.

2 Assumptions and notation

As customary, our market model is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions. Throughout the paper we will assume that there exist a money market account process paying a constant interest rate r .

Let S_t be a non-dividend-paying market asset. \tilde{S}_t will denote its time-0 discounted value $e^{-rt}S_t$. The *total realized variance* on $[0, t]$ of S_t is by definition the *quadratic variation* of the natural logarithm of S_t , that is :

$$TV_t := \langle \log S \rangle_t = \lim_{|\pi| \rightarrow 0} \sum_{t_i \in \pi} |\log S_{t_{i+1}} - \log S_{t_i}|^2. \quad (2.1)$$

The limit runs over the supremum norm of all the possible partitions π of $[t_0, t]$. The *total realized volatility* is the square root of the total realized variance. The *period realized variance* and *volatility* (or realized variance/volatility *tout court*) are given respectively by $RV_t = TV_t/t$ and $\sqrt{RV_t}$ (compare with (1.2)). If X_t is a semimartingale, by taking the limit in (2.1) it is an easy check that:

$$\langle X \rangle_t = X_t^2 - 2 \int_0^t X_u - dX_u. \quad (2.2)$$

The scalar product between vectors is throughout indicated by multiplying on the left with the transposed vector \cdot^T . If J is an absolutely continuous random variable, we denote by $f_J(x)$ its probability density function and by $\phi_J(z)$ the *characteristic function*

$$\phi_J(z) := \mathbb{E}[e^{iz^T J}]. \quad (2.3)$$

For a Fourier-integrable function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ its Fourier transform will be denoted \hat{f} . For a complex-valued function or a complex plane subset, \cdot^* indicates the complex conjugate function or set.

Unless otherwise stated, when we say that a process is a martingale we mean a martingale with respect to its natural filtration.

The notation for the conditional expectation of a stochastic process X_t at time $t_0 < t$ is $\mathbb{E}_{t_0}[\cdot]$. When the distribution of a process X_t depends on other state variables x_t (as in the case of a Markov process) those are implicitly understood to be given at time t_0 by x_{t_0} . If X_t is a process admitting conditional laws, the space of the integrable functions in the t_0 -conditional distribution of X_t at time $t_0 < t$ is indicated $L_{t_0}^1(X_t)$. Equalities are always understood to hold modulo almost-sure equivalence.

If X_t is an n -dimensional Lévy process, the *characteristic exponent* of X_t is the complex-valued function $\psi_X : \mathbb{C}^n \rightarrow \mathbb{C}$ such that:

$$\mathbb{E}[e^{i\theta^T X_t}] = e^{t\psi_X(\theta)} \quad (2.4)$$

where θ ranges in the subset of \mathbb{C}^n where the left hand side is a finite quantity.

For a given choice of a *truncation function* $\epsilon(x)$ (that is, a bounded function which is $O(|x|)$ around 0) the characteristic exponent has the unique *Lévy-Khintchine* representation:

$$\psi_X(\theta) = i\mu_\epsilon^T \theta - \frac{\theta^T \Sigma \theta}{2} + \int_{\mathbb{R}^n} (e^{i\theta^T x} - 1 - i\theta^T \epsilon(x)) \nu(dx), \quad (2.5)$$

where $\mu_\epsilon \in \mathbb{R}^n$, Σ is a non-negative definite $n \times n$ matrix with real entries, and $\nu(dx)$ a Radon measure on \mathbb{R}^n having a density function that is integrable at $+\infty$ and $O(|x|^2)$ around 0. We shall normally make the standard choice $\epsilon(x) = x\mathbb{I}_{|x| \leq 1}$ and drop the dependence of μ on ϵ . (μ, Σ, ν) is then called the *characteristic triplet* or the *Lévy characteristics* of X_t .

A *stochastic time change* T_t is an \mathcal{F}_t -adapted càdlàg process, increasing and almost-surely finite, such that T_t is an \mathcal{F}_t -adapted stopping time for each t . A *time change* of an n -dimensional Lévy process X_t according to T_t is the \mathcal{F}_{T_t} -adapted process $Y_t := X_{T_t}$.

3 Martingale relations and asset price dynamics

The aim of this section is to devise a complex exponential martingale structure naturally associated to a DTC process. This construct serves to a twofold purpose. In first place it allows for a DTC-based asset price evolution enjoying the martingale property. According to general theory, this in turn enables to postulate the existence of a risk-neutral measure that correctly prices the market securities. Secondly, it defines a class of complex measure change martingales pivotal to the characteristic function representation of the the next section.

Let \mathcal{B} the vector space of the \mathcal{F}_t -supported Brownian motions with drift starting at 0:

$$\mathcal{B} = \{\mu t + \sigma W_t, \mu \in \mathbb{R}, \sigma > 0 \mid W_t \text{ is an } \mathcal{F}_t\text{-adapted Brownian motion and } X_0 = 0\}, \quad (3.1)$$

and \mathcal{J} the vector space of the \mathcal{F}_t -supported *Lévy pure jump processes* starting at 0:

$$\mathcal{J} = \left\{ \text{càdlàg } \mathcal{F}_t\text{-adapted processes } X_t \text{ with stationary indep. increments} \mid X_t = \sum_{s < t} (X_s - X_{s-}), X_0 = 0 \right\}. \quad (3.2)$$

Every Lévy process X_t starting at 0 can be decomposed as a sum:

$$X_t = X_t^c + X_t^d, \quad (3.3)$$

with $X_t^c \in \mathcal{B}$ and $X_t^d \in \mathcal{J}$. We shall refer to X_t^c and X_t^d respectively as the *continuous* and *discontinuous* parts of X_t .

The processes X_t^c and X_t^d will be in the following subject to time changes. Time changes are very general objects, so we have to introduce some additional requirements in order for our discussion to proceed towards the construction of a martingale suitable for financial modeling. One property we shall assume throughout is *continuity with respect to the time change*.

Definition 3.1. Let T_t be a time change on a filtration \mathcal{F}_t . An \mathcal{F}_t -adapted process X_t is said to be T_t -continuous¹ if it is almost-surely constant on all the sets $[T_t^-, T_t]$.

Obviously, a sufficient condition for T_t -continuity is almost-sure continuity of T_t . In particular a continuous process is T_t -continuous if and only if T_t is almost-surely continuous. Hence, of particular relevance is the class of the *absolutely continuous* time changes, with respect to which every stochastic process is continuous. Given a pair of *instantaneous rate of activity* processes, that is, two exogenously-given càdlàg positive stochastic processes (v_t, u_t) , valid time changes are given by the pathwise integrals:

$$T_t = \int_0^t v_{s-} ds, \quad (3.4)$$

$$U_t = \int_0^t u_{s-} ds. \quad (3.5)$$

The processes v_t and u_t describe the instantaneous impact of market trading on S_t at time t , thus providing a sound mathematical description of “business activity” over time.

We are now ready for the main definition. A decoupled time change of a Lévy process is the sum of the (ordinary) time changes of its continuous and discontinuous parts:

Definition 3.2. Let X_t be a Lévy process and T_t, U_t two time changes such that T_t is continuous and X_t^d is U_t -continuous. Then:

$$X_{T,U} = X_{T_t}^c + X_{U_t}^d \quad (3.6)$$

is the *decoupled time change* of X_t according to T_t and U_t .

By Jacod [23], Corollaire 10.12, a first important property of $X_{T,U}$ is that it is an $\mathcal{F}_{T_t \wedge U_t}$ -semimartingale. To avoid degenerate cases, in what follows we always assume T_t and U_t to be such that $X_{T_t}^c$ and $X_{U_t}^d$ are Markov processes².

We are now ready to define the class of exponential martingales $M_t(\theta)$ canonically associated with $X_{T,U}$. The following Proposition represents the main theoretical tool of this paper.

Proposition 3.3. Let X_t^1 be an n -dimensional Brownian Motion with drift and X_t^2 a pure jump Lévy process in \mathbb{R}^n . Let T_t^1 and T_t^2 be two time changes such that X_t^1 and X_t^2 are respectively T_t^1 and T_t^2 -continuous. Set $X_t = X_t^1 + X_t^2$ and $T_t = (T_t^1, T_t^2)$; define $X_{T_t} := X_{T_t^1}^1 + X_{T_t^2}^2$ and denote by Θ be the domain of definition of $\mathbb{E}[\exp(i\theta^T X_t^2)]$. The process:

$$M_t(\theta, X_t, T_t) = \exp(i\theta^T X_{T_t} - (T_t^1 \psi_{X^1}(\theta) + T_t^2 \psi_{X^2}(\theta))) \quad (3.7)$$

is a local martingale, and it is a martingale if and only if $\theta \in \Theta_0$, where:

$$\Theta_0 = \{\theta \in \Theta \text{ such that } \mathbb{E}[M_t(\theta, X_t, T_t)] = 1, \forall t \geq 0\}. \quad (3.8)$$

For $T_t^1 = T_t^2$, M_t reduces to the process in [8], Theorem 1, equation (8). It is important to understand that even in this simpler case Proposition 3.3 is not a consequence of Doob’s optional sampling Theorem (unlike stated in [8], Lemma 1) applied to the martingale $Z_t(\theta) = \exp(i\theta^T X_t - t\psi_X(\theta))$, because the latter is not necessarily uniformly integrable. Indeed, time-transforming a process always preserves the semimartingale property, but the martingale property is maintained for uniformly integrable martingales only, as the following important counterexample due to C. Sin shows. Choose X_t to be a Brownian motion and let $dT_t = Y_t^{2\alpha} dt$, $\alpha > 0$, where Y_t is the solution of the SDE $dY_t = Y_t dX_t$, $X_0 = 1$. It is proved in [40] that X_{T_t} , i.e. the exponential martingale solving $dZ_t = Y_t^\alpha Z_t dX_t$, is a strict local martingale³. Surprising as it may be, this means that the set Θ_0 may very well trivialize to $\{0\}$. That is, some choices of time changes may be inherently unsuitable for time-changed asset price modeling.

Hence, condition (3.8) is of critical importance to avoid degeneracies; clearly, Θ_0 can be strictly smaller than the whole Θ , thus care must be exercised when parametrizing objects of the form (3.7), price processes in particular (see equation (3.9) further on). Compare also with the interesting examples found in [31], subsection 3.8. To give a more familiar flavour to (3.8), observe that in the simple case of X_{T_t} being a one-dimensional Brownian integral, sufficient requirements for it to be satisfied are the classic *Novikov* and *Kazamaki* conditions ([39], [28]). The set Θ_0 is called the *natural parameter set*.

Having obtained martingale relations for a stochastic exponential involving $X_{T,U}$, the risk-neutral dynamics for a DTC Lévy-driven asset are given by the obvious choice. Given a one-dimensional Lévy process X_t and a decoupled time change (T_t, U_t) , we set a price observation at time $t > 0$ of spot value S_0 to be:

$$S_t = S_0 \exp(rt + i\theta_0 X_{T,U} - T_t \psi_X^c(\theta_0) - U_t \psi_X^d(\theta_0)) = S_0 e^{rt} M_t(\theta_0, X_t^c + X_t^d, (U_t, T_t)) \quad (3.9)$$

with $\theta_0 \in \Theta_0$ being such that (3.9) is real. By Proposition 3.3 we see that $e^{-rt} S_t$ is a martingale, so that S_t is a consistent price process. This is the fundamental asset model we shall use in the rest of the paper.

4 Characteristic functions and the leverage-neutral measure

We shall now study a characteristic function associated with the price process in (3.9). As mentioned in the introduction, characteristic functions of price processes are an essential mathematical object for contingent claim pricing. Compared to the standing literature our framework has two specific aspects. Firstly, the characteristic function we shall consider is not that of the (discounted) log-price alone, but it will also incorporate the quadratic variation of the log-process. That is, if \tilde{S}_t is the value of an asset discounted by r , what we are interested in is the *joint* characteristic function of $\log(\tilde{S}_t/S_0)$ and TV_t . Just as the characteristic function of the log-price allows for the derivation of pricing formulae for contingent claims $F(S_t)$, such a multivariate extension permits valuations of payoffs of the form $F(S_t, TV_t)$.

In second place, in our setup we are in presence of a time-changed structure, in both the log-price *and* (by Lemma 4.1 further on) its quadratic variation. As pointed out in [8], Theorem 1, there exists a simple expression for the characteristic function of a time-changed Lévy process X_{T_t} in terms of a Laplace transform of T_t , whose argument depends on the Lévy structure of X_t . Unlike the case of simple Lévy subordination, such a Laplace transform is not calculated in the original distribution of T_t , but entails a change of measure which effectively captures the correlation between the underlying Lévy process and the time rescaling. As it turns out, thanks to Proposition 3.3, this argument makes perfect sense also in the case of a DTC Lévy process $X_{T,U}$.

In order to even start the discussion, we must first make sure that the quadratic variation respects the additivity and time-changed structure of $X_{T,U}$. We have the following “linearity/commutativity property”:

Lemma 4.1. *A decoupled time-changed Lévy process $X_{T,U}$ satisfies:*

$$\langle X_{T,U} \rangle_t = \langle X^c \rangle_{T_t} + \langle X^d \rangle_{U_t} = \sigma^2 T_t + \langle X^d \rangle_{U_t}. \quad (4.1)$$

That is, the quadratic variation of $X_{T,U}$ is the the sum of the time changes of the quadratic variations of its continuous and discontinuous parts.

The T_t and U_t -continuity property of the underlying Lévy components plays here an essential role. Without such an assumption, this Lemma would simply be false: see Kobayashi, [29], Example 2.5.

Since T_t and U_t are of finite variation, the total realized variance of an asset as in (3.9) satisfies $TV_t = -\theta_0^2 \langle X_{T,U} \rangle_t$, and as a consequence of Lemma 4.1 we have:

$$TV_t = -\theta_0^2 (\sigma^2 T_t + \langle X^d \rangle_{U_t}). \quad (4.2)$$

Now, assume that S_t is given by (3.9) with X_t having Lévy characteristics (μ, σ^2, ν) . We build a two-parameters family of complex-valued change of measure martingales associated with $X_{T,U}$ in the following way. Define the 2-dimensional processes $C_t = (X_t^c, 0)$ and $D_t = (X_t^d, i\theta_0 \langle X \rangle_t^d)$. Application of Proposition 3.3 to $C_t + D_t$ guarantees that the \mathbb{P} -equivalent measure $\mathbb{Q}(z, w) \ll \mathbb{P}$ having Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}(z, w)}{d\mathbb{P}} = M_t((iz\theta_0, iw\theta_0), C_t + D_t, (T_t, U_t)), \quad (4.3)$$

is well defined for all $z, w \in \mathbb{C}$ such that $(iz\theta_0, iw\theta_0) \in \Theta_0$. By using relation (4.2) and operating the change of measure entailed by (4.3), the conditional characteristic function of $(\log(\tilde{S}_t/S_0), TV_t)$ at time $t_0 < t$ is seen to be:

$$\Phi_{t_0}(z, w) = \mathbb{E}_{t_0}[\exp(iz \log(\hat{S}_t/S_0) + iw TV_t)] \quad (4.4)$$

$$= \mathbb{E}_{t_0}[\exp(iz(i\theta_0 \langle X_{T_t}^c + X_{U_t}^d \rangle - T_t \psi_X^c(\theta_0) - U_t \psi_X^d(\theta_0)) - iw \theta_0^2 (\sigma^2 T_t + \langle X^d \rangle_{U_t}))] \quad (4.5)$$

$$= \mathbb{E}_{t_0}[\exp(i(iz\theta_0, iw\theta_0)^T (C_t + D_t) - T_t(iz\psi_X^c(\theta_0) + iw\theta_0^2 \sigma^2) - U_t iz\psi_X^d(\theta_0))] \quad (4.6)$$

$$= \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-T_t(\theta_0 \mu(z - iz) - \theta_0^2 \sigma^2 (z^2 + iz - 2iw)/2) - U_t(iz\psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0)))] \quad (4.7)$$

Here $\mathbb{E}^{\mathbb{Q}}[\cdot]_{t_0}$ refers to the conditional expectation taken with respect to the measure $\mathbb{Q}(z, w)$. Lastly, we compute the characteristic exponents appearing in (4.7). By definition we have:

$$\psi_X^d(\theta_0) = \int_{\mathbb{R}} (e^{i\theta_0 x} - 1 - i\theta_0 x \mathbb{I}_{|x| \leq 1}) \nu(dx). \quad (4.8)$$

Regarding ψ_D , by the usual properties of the quadratic variation we know that $\Delta X_t := X_t - X_{t-}$ obeys $(\Delta X_t)^2 = \Delta \langle X \rangle_t$, and that $\langle X_t \rangle$ jumps if and only if X_t does, whence:

$$\psi_D(z, w) = \log \mathbb{E} \left[\exp \left(\sum_{s < t} iz \Delta X_s^d + iw (\Delta X_s^d)^2 \right) \right]. \quad (4.9)$$

Therefore we calculate further:

$$\psi_D(z, w) = \int_{\mathbb{R}} (e^{izx + iw x^2} - 1 - i(zx + wx^2) \mathbb{1}_{|x| \leq 1}) \nu(dx). \quad (4.10)$$

Equivalently, if the Lévy density $\nu^D(dx \times dy)$ of D_t is explicitly known, we have:

$$\psi_D(z, w) = \int_{\mathbb{R}^2} (e^{izx + iwy} - 1 - i(zx + wy) \mathbb{1}_{|(x,y)| \leq 1}) \nu^D(dx \times dy). \quad (4.11)$$

The conditional joint characteristic function $\Phi_{t_0}(z, w)$ has been then completely described in terms of the characteristic triplet of $X_t = X_t^c + X_t^d$ and of the joint $\mathbb{Q}(z, w)$ -distribution of T_t and U_t . Since we are in a Markovian set-up, we can use the more compact notation:

$$\Phi_{t_0}(z, w) = \mathcal{L}_{T_t, U_t}^{\mathbb{Q}}(\theta_0 \mu(z - iz) - \theta_0^2 \sigma^2(z^2 + iz - 2iw)/2, iz \psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0)), \quad (4.12)$$

where $\mathcal{L}_{T_t, U_t}^{\mathbb{Q}}$ indicates the Laplace transform of the conditional joint distribution of T_t, U_t at time t_0 with respect to $\mathbb{Q}(z, w)$, and it is a given function of $t - t_0$ and initial states T_{t_0}, U_{t_0} . As a particular case, if T_t and U_t are independent Φ_{t_0} factorizes in a continuous and discontinuous part:

$$\Phi_{t_0}(z, w) = \Phi_{t_0}^c(z, w) \Phi_{t_0}^d(z, w), \quad (4.13)$$

where:

$$\Phi_{t_0}^c(z, w) = \mathbb{E}_{t_0}^{\mathbb{Q}_1}[\exp(-T_t(\theta_0 \mu(z - iz) - \theta_0^2 \sigma^2(z^2 + iz - 2iw)/2))] \quad (4.14)$$

$$= \mathcal{L}_{T_t}^{\mathbb{Q}_1}(\theta_0 \mu(z - iz) - \theta_0^2 \sigma^2(z^2 + iz - 2iw)/2), \quad (4.15)$$

and

$$\Phi_{t_0}^d(z, w) = \mathbb{E}_{t_0}^{\mathbb{Q}_2}[\exp(-U_t(iz \psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0)))] \quad (4.16)$$

$$= \mathcal{L}_{U_t}^{\mathbb{Q}_2}(iz \psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0)). \quad (4.17)$$

The measures \mathbb{Q}_1 and \mathbb{Q}_2 are those induced respectively by the Radon-Nikodym derivatives $M_t^1(iz\theta_0, X_t^c, T_t)$ and $M_t^2(iz\theta_0, iw\theta_0, D_t, U_t)$.

It is of interest emphasizing the meaning of the measure $\mathbb{Q}(z, w)$. Consider for a moment the special case of X_t being independent of T_t and U_t . By taking the inner conditional expectation in (4.4) with respect to $\sigma(T_t, U_t)$ and applying the usual properties, one directly obtains (4.7) with $\mathbb{Q}(z, w) = \mathbb{P}$. Therefore, whenever there is no dependence between the time changes and the underlying Lévy process, no change of measure is needed in order to extract the characteristic function of a price process in terms of the rate of activity. Thus, whenever correlation between X_t and (T_t, U_t) is present, $\mathbb{Q}(z, w)$ gives a measurement of how the leverage effect due to this correlation impacts price densities. In turn, by the well-known Girsanov's results, this change of measure can be absorbed in the \mathbb{P} -dynamics of the asset by a suitable alteration of parameters in the distributions of T_t and U_t . In accordance with [8], we shall call the measure $\mathbb{Q}(z, w)$ the *leverage-neutral measure*. Just as prices in a risky market can be equivalently computed in a risk-neutral environment according to a different price distribution, by an appropriate distributional modification valuations in presence of leverage effect can be performed in a different economy with no leverage.

We shall make direct use of the leverage-neutral measure in describing the *risk-leverage-neutral* joint characteristic function $\Phi_{t_0}(z, w)$ for some actual models.

5 Pricing and price sensitivities

The characteristic function found in section 4 serves to the purpose of devising analytical formulae for the valuation of European-type derivatives having a sufficiently regular payoff F . In order to do so we will extend the Fourier-inversion machinery introduced by Lewis in [33] to our multivariate context.

Recall that since all the involved processes are Markovian, it makes sense treating $\Phi_{t_0}(z, w)$ like a Gauss-Green integral kernel depending only on some given initial state at t_0 . We have the following 2-dimensional extension of Theorem 1 of [33]:

Theorem 5.1. *Let $Y_t = \log S_t$, with S_t given by (3.9). Let $F(x, y) \in L^1_{t_0}(Y_t, \langle Y \rangle_t)$ for all $t > t_0$, be a positive payoff function having analytical Fourier transform $\hat{F}(z, w)$ in a multi-strip $\Sigma_F = \{(z, w) \in \Theta, \alpha_1 < z < \alpha_2, \beta_1 < w < \beta_2\} \subseteq \mathbb{C}^2$. Suppose further that $\Phi_{t_0}(z, w)$ is analytical in $\Sigma_\Phi = \{(z, w) \in \Theta, \gamma_1 < z < \gamma_2, \eta_1 < w < \eta_2\}$ and that $\Phi_{t_0}(z, w) \in L^1(dz \times dw)$. If $\Sigma_F \cap \Sigma_\Phi^* \neq \emptyset$ then for every multi-line:*

$$L_{k_1, k_2} = \{(x + ik_1, y + ik_2), (x, y) \in \mathbb{R}^2\} \subset \Sigma_F \cap \Sigma_\Phi^* \quad (5.1)$$

we have that the time- t_0 value of the contingent claim F maturing at time t is:

$$\begin{aligned} \mathbb{E}_{t_0}[e^{-r(t-t_0)} F(Y_t, \langle Y \rangle_t)] &= \frac{e^{-r(t-t_0)}}{4\pi^2} \cdot \\ &\int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} e^{-iw \langle Y \rangle_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz} \Phi_{t_0}(-z, -w) \hat{F}(z, w) dz dw. \end{aligned} \quad (5.2)$$

It is clear that modifying the asset dynamics specifications only acts on Φ_{t_0} , whereas changing the claim to be priced influences \hat{F} only. Also notice that by setting either variable to 0, we are able to extract from (5.2) prices for both plain vanilla and pure volatility derivatives. In particular, this means that the classic pricing integrals in [32] and [33], are special cases of the equation above which can be recovered when F does not depend on the realized volatility and when Φ_{t_0} is either obtained from a diffusion ([32]), or a Lévy process ([33]).

In addition, this representation is particularly useful if we are interested in the sensitivities of the claim value with respect to the underlying state variables. Let us consider for instance the Delta (sensitivity with respect to the change in value of the underlying) and Gamma (sensitivity with respect to the rate of change in value of the underlying) of valuations performed through formula (5.2). Call $I(r, t_0, z, w)$ the integrand on the right hand side of (5.2); by differentiating under integral sign and noting that Φ_{t_0} bears no dependence on S_{t_0} we see that:

$$\Delta_t := \frac{\partial}{\partial S} \mathbb{E}_{t_0}[e^{-r(t-t_0)} F(Y_t, \langle Y \rangle_t)] = -\frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} \frac{iz}{S_{t_0}} I(r, t_0, z, w) dz dw, \quad (5.3)$$

and

$$\Gamma_t := \frac{\partial^2}{\partial S^2} \mathbb{E}_{t_0}[e^{-r(t-t_0)} F(Y_t, \langle Y \rangle_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} \frac{iz - z^2}{S_{t_0}^2} I(r, t_0, z, w) dz dw. \quad (5.4)$$

Mutatis mutandis we can repeat this argument if we want to determine the price sensitivity with respect to the quadratic variation $\langle Y \rangle_t$. Finally, $\Phi_{t_0}(z, w)$ could also depend on other state variables (e.g. the instantaneous rate of market activity v_{t_0}) known at time t_0 . By calling ν one such variable we have that:

$$\begin{aligned} \mathcal{V}_t := \frac{\partial}{\partial \nu} \mathbb{E}_{t_0}[e^{-r(t-t_0)} F(Y_t, \langle Y \rangle_t)] &= \frac{e^{-r(t-t_0)}}{4\pi^2} \cdot \\ &\int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} e^{-iw \langle Y \rangle_{t_0}} S_{t_0}^{-iz} e^{r(t-t_0)iz} \frac{\partial \Phi_{t_0}}{\partial \nu}(-z, -w) \hat{F}(z, w) dz dw. \end{aligned} \quad (5.5)$$

This is especially well-suited to the case in which $\Phi_{t_0}(z, w)$ is exponentially affine in ν , i.e.

$$\Phi_{t_0}(z, w) = \exp(A(z, w, t - t_0)\nu_{t_0} + B(z, w, t - t_0)), \quad (5.6)$$

when we have:

$$\frac{\partial \Phi_{t_0}}{\partial \nu}(-z, -w) = A(-z, -w, t - t_0) \Phi_{t_0}(-z, -w). \quad (5.7)$$

In the next section we find explicitly Φ_{t_0} for a number of decoupled time-changed models.

6 Specific model analysis

We now describe the underlying DTC Lévy dynamics (3.9) for some classic asset price processes, and find the corresponding risk-leverage-neutral characteristic function $\Phi_{t_0}(z, w)$. In view of the theory developed this far this accomplishes two goals. In first place, it specifies the fundamental piece of information needed to numerically implement the results of the previous section. Secondly, it provides convincing evidence that DTCs offer a natural unifying framework for *a priori* different strains of financial asset models (e.g. continuous/jump diffusions, jump diffusions with stochastic volatility, Lévy processes).

The models analyzed fall into two main categories: plain Lévy processes (Secs. 6.1, 6.3, 6.4) and decoupled time-changed finite-activity Lévy models⁴ (Secs. 6.2, 6.5, 6.6). The first group can be recovered in a straightforward way from our framework by setting $T_t, U_t = t$. The second group retains a non-trivial DTC structure and constitutes an ideal testing ground for our theory.

The approach will be progressing from the more simple model (Black-Scholes), to the more sophisticated one (Fang). However, each finite-activity DTC Lévy model can be thought of as a particular case of the model in subsection 6.6, by considering the appropriate parameters/time change degeneracies. In practice, this means that a single software implementation is able to capture every jump diffusion-type model hereby illustrated. Subcases are dealt with by just voiding the relevant parameters, as it is clearly seen in the online MATHEMATICA[®] supplement.

As one might expect, the value of θ_0 in (3.9) is implicitly given throughout as the natural $\theta_0 = -i$. The feasibility of this choice from a DTC perspective lies in the specific structure of the time changes. Indeed, T_t and U_t will always be absolutely continuous time changes with rates of activity v_t and u_t given by well-behaved diffusions admitting unique strong solutions. Essentially, this means that in the cases we cover we always have $\Theta_0 = \Theta$ in (3.8).

6.1 Black-Scholes model

The classic SDE with constant parameters σ, r driven by a Brownian motion W_t :

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (6.1)$$

can be trivially recovered from (3.9) by setting the triplet for the underlying Lévy process $X_t = X_t^c$ to be $(0, \sigma, 0)$ and letting $T_t = t, U_t = 0$, so that $X_{T,U} = X_t$. From (4.12), we have immediately:

$$\Phi_{t_0}(z, w) = \exp(-(t - t_0)\sigma^2(z^2 + iz - 2iw)/2). \quad (6.2)$$

6.2 Heston model

The model proposed by Heston in [22] to explain the leptokurtic feature of price observations is classically represented in a risk-neutral measure by the dependent pair of SDEs:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1; \quad (6.3)$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2. \quad (6.4)$$

The parameters $\alpha, \theta, \eta > 0$ need to satisfy the *Feller condition* of [18], and the Brownian motions W_t^1, W_t^2 are correlated with $\langle W_t^1, W_t^2 \rangle = \rho t$. By a well-known Theorem ([27], Ch. 3, Theorem 4.6), any continuous martingale N_t can be written as $N_t = W_{\langle N_t \rangle}$ for a certain Brownian motion W_t , which implies that the DTC structure of $X_{T,U}$ in the Heston model corresponds to a standard Brownian motion W_t time-changed by:

$$T_t = \int_0^t v_u du. \quad (6.5)$$

Since we are in presence of leverage between X_t and T_t , to find the characteristic function $\Phi_{t_0}(z, w)$ corresponding to this model we apply the complex-plane version of Girsanov's Theorem, which gives us the leverage-neutral dynamics of v_t . This results in changing the dynamics parameters α, θ of (6.4) to:

$$\alpha^z = \alpha - i\rho z\eta; \quad (6.6)$$

$$\theta^z = \alpha\theta/\alpha^z. \quad (6.7)$$

By (3.9) this determines Φ_{t_0} as:

$$\Phi_{t_0}(z, w) = \mathcal{L}_{T_t}^{\mathbb{Q}}(z^2/2 + iz/2 - iw) = \mathbb{E}_{t_0} \left[e^{-(z^2/2 + iz/2 - iw) \int_{t_0}^t v_u^z du} \right] = \mathcal{L}_{T_t}^z(z^2/2 + iz/2 - iw) \quad (6.8)$$

where v_t^z is given by the dynamics (6.4) with the parameters (6.6)-(6.7). The conditional Laplace transform of the integrated square root process $T_t^z := \int_0^t v_u^z du$ is well-known⁵ analytically (Dufresne [13]).

The case $T_t = t$ reverts back to the Black-Scholes model, when (6.8) collapses to (6.2) with $\sigma^2 = v_0$.

6.3 Jump diffusion models

In their classic works, Merton and Kou ([36], [30]) proposed to model the log-price dynamics as a finite-activity jump diffusion, which in practice has been observed to better fit the smile of option values for short maturity ranges. The risk-neutral asset dynamics are given by:

$$dS_t = rS_t dt + \sigma_t S_t dW_t + S_t J dN_t - \kappa \lambda S_t dt \quad (6.9)$$

where W_t is a standard Brownian motion, N_t is a Poisson counter of intensity λ , and J is the jump size distribution. N_t and W_t are assumed to be independent, and the compensator κ equals $\phi_J(-i) - 1$. In the Merton model J is normally distributed $J \sim \mathcal{N}(\mu, \delta^2)$, whereas Kou assumed for J an asymmetrically skewed double-exponential distribution, that is, the density function $f_J(x)$ to be of the form:

$$f_J(x) = \begin{cases} \alpha p e^{-\alpha x} & \text{if } x \geq 0 \\ \beta q e^{\beta x} & \text{if } x < 0 \end{cases} \quad (6.10)$$

for $\alpha > 1, \beta > 0$ and $p + q = 1$.

In these models no time change is involved, so $X_{T,U}$ coincides with the underlying Lévy process X_t having characteristic triplet $(0, \sigma^2, \lambda f_J(x) dx)$. Thus, to completely characterize $\Phi_{t_0}(z, w)$ all we need to do is to determine the characteristic exponents ψ_X^d and ψ_D in (4.8) and (4.10). The former is given by the standard theory as:

$$\psi_X^d(\theta) = \lambda(\phi_J(\theta) - 1). \quad (6.11)$$

In order to compute $\psi_D(z, w)$ notice that by (4.9) the process $(X_t^d, \langle X^d \rangle_t)$ is just a bivariate compound Poisson process of joint density $f_{J,J^2}(x, y)$ and intensity λ for each component. Therefore use of (4.11) gives:

$$\psi_D(z, w) = \lambda(\phi_{J,J^2}(z, w) - 1); \quad (6.12)$$

where $\phi_{J,J^2}(z, w)$ is the joint characteristic function of J and J^2 . We conclude from (4.12) that Φ_{t_0} has the exponential Lévy structure:

$$\Phi_{t_0}(z, w) = \exp((t - t_0)(\sigma^2(-z^2/2 - iz/2 + 2iw)/2 + \lambda(\phi_{J,J^2}(z, w) - iz\phi_J(-i) + iz - 1))). \quad (6.13)$$

Now for the Merton model we have, completing the square:

$$\begin{aligned} \phi_{J,J^2}(z, w) &= \int_{-\infty}^{+\infty} \frac{e^{izx + iwx^2} e^{-\frac{(x-\mu)^2}{2\delta^2}}}{\delta\sqrt{2\pi}} dx = e^{\frac{i\mu z - \sigma^2 z^2/2 + i\mu^2 w}{1 - 2i\delta^2 w}} \int_{-\infty}^{+\infty} \frac{e^{-(x\sqrt{1-2\delta^2 w} - (i\delta^2 z - \mu)/\sqrt{1-2\delta^2 w})^2/2\delta^2}}{\delta\sqrt{2\pi}} dx \\ &= \frac{\exp\left(\frac{i\mu z - \delta^2 z^2/2 + i\mu^2 w}{1 - 2i\delta^2 w}\right)}{\sqrt{1 - 2i\delta^2 w}}, \end{aligned} \quad (6.14)$$

and the integral converges for $\text{Im}(w) > -1/2\sigma^2$. For the Kou Model we can write:

$$\phi_{J,J^2}(z, w) = \phi_{J_+, J_+^2}(z, w) + \phi_{J_-, J_-^2}(z, w). \quad (6.15)$$

The characteristic function of the positive part is:

$$\begin{aligned}\phi_{J_+, J_+^2}(z, w) &= \int_0^{+\infty} e^{izx + iw x^2} \alpha p e^{-\alpha x} dx = \alpha p e^{-\frac{(\alpha - iz)^2}{4iw}} \int_0^{+\infty} e^{-(\sqrt{-iw}x + (\alpha - iz)/2\sqrt{-iw})^2} dx \\ &= \alpha p \sqrt{\pi} e^{-\frac{(\alpha - iz)^2}{4iw}} \left(\frac{\operatorname{Erfc}\left(\frac{\alpha - iz}{2\sqrt{-iw}}\right)}{2\sqrt{-iw}} \right),\end{aligned}\quad (6.16)$$

which converges for $\operatorname{Im}(w) > 0$. The negative part is obtained by a similar calculation.

6.4 Tempered Lévy stable and CGMY

Another way of obtaining Lévy distributions for the asset price is that of directly specifying an infinite activity Lévy measure $\nu(dx)$. In this case the asset evolution may be thought of as being solely driven by the infinitely often occurring discontinuities of the process, which is therefore going to be of pure jump type plus a deterministic compensator. In our language we can represent the underlying stochastics of this type of processes as $X_{T,U} = X_t = X_t^d$, with X_t being a pure jump Lévy process of Lévy measure $\nu(dx)$.

Two instances are the tempered Lévy stable process and the CGMY model. Both of these are obtained as an exponential smoothing of a stable-type distribution; the second can be viewed as a generalization of the first allowing for an asymmetrical skewing between the distribution of positive and negative jumps. The Lévy density for a CGMY process is:

$$\frac{d\nu(x)}{dx} = \frac{c_- e^{-\beta_- |x|}}{|x|^{1+\alpha_-}} \mathbb{I}_{\{x < 0\}} + \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}} \mathbb{I}_{\{x \geq 0\}}. \quad (6.17)$$

Which is well defined for all $c_+, c_-, \beta_+, \beta_- > 0$, $\alpha_+, \alpha_- < 2$. When $\alpha_+ = \alpha_-$ one has the tempered stable process. For simplicity in what follows we assume $\alpha_+, \alpha_- \neq 0, 1$; for such values the involved characteristic functions still exist, but lead to particular cases.

As in subsection 6.3, to fully characterize $\Phi_{t_0}(z, w)$ we only need to determine $\psi_X^d(\theta)$ and $\psi_D(z, w)$. Once this is done then (4.12) simplifies to yield yet another Lévy exponential structure:

$$\Phi_{t_0}(z, w) = \exp((t - t_0)(\psi_D(z, w) - iz\psi_X^d(-i))). \quad (6.18)$$

By setting $\gamma_1 = \int_{-1}^1 x d\nu(x)$, the exponent $\psi_X^d(\theta)$ is given ([10], Proposition 4.2) as:

$$\psi_X^d(\theta) = \gamma_1 + \Gamma(-\alpha_+) \beta_+^{\alpha_+} c_+ \left(\left(1 - \frac{i\theta}{\beta_+}\right)^{\alpha_+} - 1 + \frac{i\theta\alpha_+}{\beta_+} \right) + \Gamma(-\alpha_-) \beta_-^{\alpha_-} c_- \left(\left(1 + \frac{i\theta}{\beta_-}\right)^{\alpha_-} - 1 - \frac{i\theta\alpha_-}{\beta_-} \right). \quad (6.19)$$

Set $\gamma_2 = \int_{-1}^1 x^2 d\nu(x)$; MATHEMATICA[®] finds the following interesting representation for the positive part ψ_D^+ of ψ_D :

$$\psi_D^+(z, w) = iz\gamma_1 + iw\gamma_2 + \int_0^{+\infty} (e^{izx + iw x^2} - 1 - (izx + iw x^2)) \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}} dx = \quad (6.20)$$

$$\begin{aligned} & iz\gamma_1 + iw\gamma_2 + ic_+ \beta_+^{\alpha_+} \left(-w \frac{\Gamma(2 - \alpha_+)}{2i\beta_+^2} - z \frac{\Gamma(1 - \alpha_+)}{2i\beta_+} + i\Gamma(-\alpha_+) \right) - c_+ (\beta_+ - iz)^{\alpha_+} \left(\frac{i(\beta_+ - iz)^2}{w} \right)^{-\alpha_+/2} \\ & \left(\sqrt{\frac{i(\beta_+ - iz)}{w}} \Gamma\left(\frac{1}{2} - \frac{\alpha_+}{2}\right) {}_1F_1\left[\frac{1 - \alpha_+}{2}, \frac{3}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] - \Gamma\left(-\frac{\alpha_+}{2}\right) {}_1F_1\left[-\frac{\alpha_+}{2}, \frac{1}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] \right). \end{aligned} \quad (6.21)$$

Here Γ is the Euler Gamma and ${}_1F_1$ the confluent hypergeometric function. The strip of convergence of (6.21) is the set $\Sigma_\Phi = \{(z, w), \operatorname{Im}(w) > 0, \operatorname{Im}(z) > -\beta_+\}$. The determination ψ_D^- has a similar expression.

6.5 Bates model

The Bates model is a widely known jump diffusion/stochastic volatility model introduced in [2] and [3] to value foreign exchange options. In the present discussion, it provides us with a first instance of decoupled time change not

otherwise obtainable as an ordinary time change. Specifically, the Bates model is a DTC model with a time-changed continuous part and a time-homogeneous jump part.

The dynamics for the asset price and the stochastic volatility are driven by two correlated Brownian motions W_t^1 and W_t^2 , and a Poisson process N_t of parameter λ , combining to the SDEs:

$$dS_t = rS_{t-} + \sqrt{v_t}S_{t-}dW_t^1 + S_{t-}JdN_t - \kappa\lambda S_{t-}dt; \quad (6.22)$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2. \quad (6.23)$$

The usual choice for the distribution of J is normal $\mathcal{N}(\log(1 + \kappa) - \delta^2/2, \delta^2)$, so that κ here represents the average percentage jump ratio. The dependence structure and the requirements on the parameters of the driving Brownian and Poisson processes are the same as jointly the Heston and Merton models. The Bates model is in effect a combination of the two.

The underlying DTC structure of the Bates model is given by $X_{T,U} = X_{T_t}^c + X_t^d$ with the characteristic triplet for X_t being $(0, 1, \lambda f_J(x)dx)$. The time change of the continuous part is, as in subsection 6.2:

$$T_t = \int_0^t v_u du. \quad (6.24)$$

It is straightforward to see that $\Phi_{t_0}(z, w)$ decomposes as:

$$\Phi_{t_0}(z, w) = \Phi_{t_0}^c(z, w)\Phi_{t_0}^d(z, w), \quad (6.25)$$

where $\Phi_{t_0}^c(z, w)$ and $\Phi_{t_0}^d(z, w)$ are given respectively by (6.8) and (6.13) (with $\sigma = 0$). Thus:

$$\Phi_{t_0}(z, w) = \mathbb{E}_{t_0} \left[e^{-(z^2/2 + iz/2 - iw) \int_{t_0}^t v_u^z du} \right] \exp((t - t_0)\lambda(\phi_{J,J^2}(z, w) - iz\kappa - 1)) \quad (6.26)$$

$$= \mathcal{L}_{T_t}^z(z^2/2 + iz/2 - iw) \exp((t - t_0)\lambda(\phi_{J,J^2}(z, w) - iz\kappa - 1)). \quad (6.27)$$

Expression (6.27) is of a completely new kind. Until now we have encountered either exponential Lévy models, or exponential affine functions arising as solutions of a PDE problem. Here we have a mixture of the two: a time-homogeneous jump factor, modeled as a compound Poisson process and a continuous diffusion factor, whose characteristic function solves a time-inhomogeneous diffusion problem. Once again we remark the degenerate case $T_t = t$, which just yields a Merton jump diffusion with diffusion coefficient $\sqrt{v_0}$.

6.6 Fang Model

A relatively little known model has been suggested and empirically studied by H. Fang in [17]. The asset price dynamics consist of a jump diffusion with an Heston-type stochastic volatility and a *stochastic jump rate*, itself modeled as a square root process. The risk-neutral dynamics are:

$$dS_t = rS_{t-}dt + \sqrt{v_t}S_{t-}dW_t^1 + S_{t-}JdN_t - \kappa\lambda_t S_{t-}dt; \quad (6.28)$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2; \quad (6.29)$$

$$d\lambda_t = \alpha_\lambda(\theta_\lambda - \lambda_t)dt + \eta_\lambda\sqrt{\lambda_t}dW_t^3. \quad (6.30)$$

Both of the diffusion parameter sets must obey Feller's condition. As usual $\langle W_t^1, W_t^2 \rangle = \rho dt$. Like in Bates model the jumps J are normal with mean $\log(1 + \kappa) - \delta^2/2$ and variance $\delta^2/2$. Conditional on λ_t , N_t is a Poisson process of jump parameter λ_t independent of every other process. The Brownian motion W_t^3 is also assumed to be independent of all the other random variables⁶.

This model has a clear DTC Lévy structure $X_{T,U}$ given by $T_t = \int_0^t v_u du$, $U_t = \int_0^t \lambda_u du$, X_t having characteristic triplet $(0, 1, f_J(x)dx)$. In the decomposition (4.13) we immediately recognize $\Phi_{t_0}^c$ as the risk-leverage-neutral characteristic function of an Heston process of variance v_t , and $\Phi_{t_0}^d$ as that of a compound Poisson process time-changed with U_t . The function Φ_{t_0} is then given by:

$$\Phi_{t_0}(z, w) = \Phi_{t_0}^c(z, w)\Phi_{t_0}^d(z, w) = \mathcal{L}_{T_t}^z(z^2/2 + iz/2 - iw)\mathcal{L}_{U_t}(iz\kappa - \phi_{J,J^2}(z, w) + 1). \quad (6.31)$$

The Laplace transforms of the integrated-square root processes arising from v_t^z and λ_t are known, and the leverage-neutral version v_t^z of v_t has been given in 6.2. Observe that there is no leverage effect in the jump part because of the assumptions on W_t^3 . Finally, notice that the case $U_t = t$ is just the Bates model where the jump activity rate equals λ_0 .

7 Numerical testing and final remarks

7.1 Implementation for the target volatility option

For validation purposes we numerically implemented the pricing formula (5.2) on MATHEMATICA[®] for various models, and compared the analytical prices found to a MATLAB[®] simulation following a naïve Euler scheme.

The target volatility call option is a natural candidate for such testing, being to our knowledge one instance of a joint asset/volatility derivatives that has actually been traded. As mentioned in the introduction, a TVO is a contingent claim based on underlying asset S paying at the expiration t the amount:

$$F(S_t, RV_t) = \frac{\bar{\sigma}}{\sqrt{RV_t}}(S_t - K)^+, \quad (7.1)$$

for a strike price K and a *target volatility* level $\bar{\sigma}$ written in the contract. Clearly, the payoff of a TVO will be greater than that of the corresponding vanilla call, as long as the volatility realized by the underlying is lower than the target volatility. It can be shown (Di Graziano and Torricelli, [12]) that an at-the-money TVO is priced, up to a second order error, by the at-the-money Black-Scholes call value of implied volatility $\bar{\sigma}$. Thus in some sense $\bar{\sigma}$ represents the volatility prediction an investor attaches to a call option at present time, as opposed to the current market implied volatility values. If the latter are too high he may choose to trade in a TVO instead.

The Fourier transform $\hat{F}(z, w)$ of a TVO payoff to be used in (5.2) is:

$$\hat{F}(z, w) = \bar{\sigma}(1 + i)\sqrt{\frac{\pi t}{2w}} \frac{K^{1+iz}}{(iz - z^2)}, \quad (7.2)$$

which exists and is analytical in the strip $\Sigma_F = \{(z, w) \in \mathbb{C}^2, \text{Im}(z) > 1, \text{Im}(w) > 0\}$.

The model parameters have been taken from the S&P 500 fits of [17] and are illustrated in table 1. Tables 2 to 5 summarize the result obtained for four different sets of market and contract conditions. We observe homogeneous values for the option prices across the various models for any given data set. Also, an excellent match is exhibited with the benchmark Monte Carlo simulation. For other fast and efficient pricing formulae specific to the TVO also refer to [12].

Actual computations can be found in the MATHEMATICA[®] notebook accompanying this paper. A powerful feature of the DTC approach can be thereby appreciated. As highlighted in section 6, by voiding the relevant parameters, a single coding for equation (6.31) is enough to produce prices for several different models.

7.2 Conclusion

In this paper we have introduced the concept of decoupled time change, with the intent of accommodating in an unitary time change framework models escaping the classic definition of time-changed Lévy process. Most notably, Bates stochastic volatility/jump diffusion model and Fang's stochastic volatility/stochastic jump rate model are seen to have their own, previously unrecognised, time re-scaled Lévy structure.

We obtained martingale relations for stochastic exponentials based on DTC Lévy processes, and built up asset price dynamics based on such exponential martingales. In order to do so, we had to resort to the semimartingale representation of $X_{T,U}$ as given by its local characteristics.

Inspired by [8], we proceeded to find a relation between the joint characteristic function of the log-price dynamics /quadratic variation and the joint Laplace transform of the time changes. We used this as an integral kernel for pricing derivative securities paying off on an asset and/or the volatility it accrues before maturity. In doing so, the idea of

leverage-neutral measure and its action on a distribution parameters has been recalled and extended to the present set-up.

Several stochastic models have been analyzed in the DTC framework. In the accounted cases we outlined the underlying DTC structure and found the risk-leverage-neutral characteristic function. For numerical comparison and validation, we focused on one particular joint asset/volatility derivative, the target volatility option. One important contribution from a computational standpoint, is that a single software implementation can output prices for several different models.

8 Appendix: proofs

The proof of the core Proposition (3.3) is based on the theory of martingale representations. In particular, Jacod and Shiryaev ([24], Ch. 1, 2) Kuchler and Sorensen ([31], Ch. 11) and Jacod ([23], Ch. X), provide the necessary tools for understanding the key results. As far as this author is aware, this type of approach is relatively unused in the financial field, so we briefly illustrate the fundamental concepts.

Let us introduce the notion of *local characteristics* of a semimartingale, that play in a general semimartingale representation the analogous role of the Lévy characteristics. Define the *Doléans-Dade exponential* of an n -dimensional process X_t starting at 0 as:

$$\mathcal{E}(X_t) = e^{X_t - \langle X_t^c \rangle / 2} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \quad (8.1)$$

where X_t^c denotes the continuous part of X_t and the infinite product converges uniformly. This is known to be the solution of the SDE $dY_t = Y_{t-} dX_t$, $Y_0 = 1$.

Next, let $\epsilon(x)$ be a truncation function and $(\alpha_t, \beta_t, \rho(dt \times dx))$ be a triplet of predictable processes respecting the following properties (compare [24], Ch. 2, 2.12-2.14):

- P1. α_t has finite variation and values in \mathbb{R}^n ;
- P2. β_t is an $n \times n$ matrix-valued continuous process of finite variation such that $\beta_t - \beta_s$ is symmetric positive definite almost surely for $s < t$;
- P3. $\rho(dx \times dt)$ is a random measure on $\mathbb{R}_+ \times \mathbb{R}^n$, almost-surely integrable at $+\infty$ and $O(|x|^2)$ around 0, having the property that $\rho(\{t\} \times \mathbb{R}^n) \leq 1$;
- P4. $\Delta \alpha_t = \int_{\mathbb{R}^n} \epsilon(x) \rho(\{t\} \times dx)$.

For $\theta \in \mathbb{C}^n$, associate with $(\alpha_t, \beta_t, \rho(dt \times dx))$ the following complex-valued functional:

$$\Psi_t(\theta) = i\theta^T \alpha_t - \theta^T \beta_t \theta / 2 + \int_0^t \int_{\mathbb{R}^n} (e^{i\theta^T x} - 1 - i\theta^T x \epsilon(x)) \rho(ds \times dx). \quad (8.2)$$

This functional is well defined on:

$$\mathcal{D} = \left\{ \theta \in \mathbb{C}^n \text{ such that } \int_0^t \int_{\mathbb{R}^n} e^{i\theta^T x} \epsilon(x) \rho(ds \times dx) < +\infty \text{ almost surely} \right\}, \quad (8.3)$$

and because of the assumptions above it is also predictable and of finite variation. The local characteristics of a semimartingale are then defined as follows:

Definition 8.1. Let X_t be an n -dimensional semimartingale supported on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and let $\mathcal{D} \subset \mathbb{C}^n$ be like in (8.3). Additionally assume that $\mathcal{E}(\Psi_t(\theta)) \neq 0$ for all $\theta \in \mathcal{D}$. The *local characteristics* of X_t are the unique predictable processes $(\alpha_t, \beta_t, \rho(dx \times dt))$ satisfying P1, P2, P3, P4 and such that $\exp(i\theta^T X_t) / \mathcal{E}(\Psi_t(\theta))$ is a local martingale for all $\theta \in \mathcal{D}$.

This definition is not the most general, (the condition $\mathcal{E}(\Psi_t(\theta)) \neq 0$ can be relaxed) but it is the most useful in practice. As we will see, the *cumulant process* $\Psi_t(\theta)$ plays the role of a “stochastic compensator” for $i\theta^T X_t$. The measure ρ is the compensator of the jump measure of X_t and it is 0 if and only if X_t is continuous. Moreover, X_t is

a Lévy process if and only if its local characteristics are deterministic and of the form $(t\alpha, t\beta, \nu dt)$ for the given Lévy triplet (α, β, ν) , in which case $\Psi_t(\theta)$ coincides with $t\psi_X(\theta)$.

We now explain how the property of continuity with respect to a time change interacts with the local characteristics. Indeed, if the condition in Definition 3.1 is met by the involved processes, the local characteristic of a time-changed semimartingale are well-behaved, in the sense of the next Theorem. Recall that if \mathcal{B} is a Borel space, the time change of a random measure $\rho(dt \times dx)$ on the product measure space $\Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n)$ is the random measure defined by:

$$\rho(dT_t \times dx)(\omega, t, B) := \rho([0, T_t(\omega)] \times B)(\omega) \quad (8.4)$$

for all $\omega \in \Omega$, $t \geq 0$ and sets $B \in \mathcal{B}(\mathbb{R}^n)$.

Theorem A. *Let X_t be a semimartingale having local characteristics $(\alpha_t, \beta_t, \rho(dx \times dt))$ and cumulant process $\Psi_t^X(\theta)$ with domain \mathcal{D} , and let T_t be a time change such that X_t is T_t -continuous. Then the time-changed semimartingale $Y_t = X_{T_t}$ has local characteristics $(\alpha_{T_t}, \beta_{T_t}, \rho(dT_t \times dx))$ and the cumulant process $\Psi_t^Y(\theta)$ equals $\Psi_{T_t}^X(\theta)$, for all $\theta \in \mathcal{D}$.*

Proof. Kallsen and Shiryaev, [26], Lemma 2.7. \square

Another important consequence of Definition (3.1) is the following:

Theorem B. *Let T_t be a time change with respect to some filtration \mathcal{F}_t and let X_t be a T_t -continuous semimartingale. For all \mathcal{F}_t -predictable integrands H_t , we have that H_{T_t} is \mathcal{F}_{T_t} -predictable, and:*

$$\int_0^{T_t} H_s dX_s = \int_0^t H_{T_s-} dX_{T_s}. \quad (8.5)$$

This is particularly useful when trying to compute the quadratic variation of a time-changed process. Finally, we need a result on linear transformation of a semimartingale and the corresponding change in the local characteristics.

Theorem C. *Let X_t be an n -dimensional semimartingale of local characteristics $(\alpha_t, \beta_t, \rho(dx \times dt))$ and let M be a linear transformation from \mathbb{R}^n to \mathbb{R}^m . Then $Y_t = MX_t$ is an m -dimensional semimartingale whose local characteristics $(\alpha_t^Y, \beta_t^Y, \rho^Y(dx \times dt))$ are given by:*

$$\begin{aligned} \alpha_t^Y &= M\alpha_t + \int_0^t \int_{\mathbb{R}^n} (\epsilon(Mx) - M\epsilon(x)) \rho(dx \times ds) \\ \beta_t^Y &= M\beta_t M^T \\ \rho^Y(B \times [0, t]) &= \rho(\{x \in \mathbb{R}^n | Mx \in B\} \times [0, t]), \quad \forall B \in \mathcal{B}(\mathbb{R}^m). \end{aligned} \quad (8.6)$$

Proof. Eberlein, Papapantoleon and Shiryaev, [15], Proposition 2.4. \square

Having at hand these theoretical instruments, we can now proceed to prove Proposition 3.3.

Proof of Proposition 3.3. Let $(\mu, \Sigma, 0)$ and $(0, 0, \nu)$ be the Lévy triplets of X_t^1 and X_t^2 . Because of the T_t^1 and T_t^2 -continuity assumption, we can apply Theorem A and we immediately have that the local characteristics of $X_{T_t^1}^1$ and $X_{T_t^2}^2$ are respectively $(T_t^1\mu, T_t^1\Sigma, 0)$ and $(0, 0, dT_t^2\nu)$.

Now consider the $2n$ -dimensional martingale $Y_t = \begin{pmatrix} X_{T_t^1}^1 \\ X_{T_t^2}^2 \end{pmatrix}^T$. Clearly, the random measure ρ^Y from the characteristics of Y_t is supported on the last set of n variables, and its margin on such subspace coincides with $dT_t^2\nu$. Hence, Theorem C applied to Y_t , with M being the $n \times 2n$ matrix made by the juxtaposition of two $n \times n$ identity blocks, implies that the local characteristics of X_{T_t} are $(T_t^1\mu, T_t^1\Sigma, \nu dT_t^2)$.

Now, observe that for every $\theta \in \Theta$ the exponential $\mathcal{E}(\Psi_t(\theta))$ is well-defined. Moreover, by [24], Proposition 2.9, (i) and [15], Lemma 2.6, X_{T_t} is *quasi-left-continuous*⁷, which implies that $\Psi_t(\theta)$ has no atoms on the sets $\{t\} \times dx$. Consequently, $\Delta\Psi_t(\theta) = 0$; also, since Ψ_t is of finite variation, we have $\langle \Psi^c \rangle_t = 0$. Hence:

$$\mathcal{E}(\Psi_t(\theta)) = \exp(\Psi_t(\theta)). \quad (8.7)$$

In particular this means that $\mathcal{E}(\Psi_t(\theta))$ never vanishes. By using the definition of local characteristics we then see that the stochastic exponential $M_t(\theta, X_t, T_t)$ is a local martingale for all $\theta \in \Theta$. But a positive local martingale is a supermartingale, and a supermartingale is a martingale if and only if it has constant expectation. Therefore $M_t(\theta, X_t, T_t)$ is a martingale if and only if $\theta \in \Theta_0$. \square

The proof of Lemma 4.1 is a mere application of Theorem B.

Proof of Lemma 4.1. Clearly:

$$\langle X_{T,U} \rangle_t = \langle X_T^c \rangle_t + \langle X_U^d \rangle_t + 2\langle X_T^c, X_U^d \rangle_t. \quad (8.8)$$

It is clear from the Proposition 3.3 that X_T^c is a continuous process whereas X_U^d is of pure jump-type. Therefore we have $\langle X_T^c, X_U^d \rangle_t = 0$. Since $\langle X^c \rangle_t = \sigma^2$, the result directly follows from Theorem A, applied to $\langle X_T^c \rangle_t$ and $\langle X_U^d \rangle_t$. \square

We conclude by proving the pricing equation of section 5. The proof by Lewis ([33], Theorem 3.2, Lemma 3.3 and Theorem 3.4) can be easily extended to include dependence of the payoff on $\langle Y \rangle_t$.

Proof of Theorem 5.1. By writing the expectation as an inverse-Fourier integral (which can be done by the assumptions on F and because Φ_{t_0} is a characteristic function) and passing the expectation under the integration sign we have:

$$\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t, \langle Y \rangle_t)] = \mathbb{E}_{t_0} \left[\frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} S_t^{-iz} e^{-iw\langle Y \rangle_t} \hat{F}(z, w) dz dw \right] \quad (8.9)$$

$$= \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} e^{-iw\langle Y \rangle_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz} \Phi_{t_0}(-z, -w) \hat{F}(z, w) dz dw. \quad (8.10)$$

The only thing to prove is that Fubini's Theorem application is justified. Let $N_t = M_t(\theta_0, X_t, (T_t, U_t))$ be the discounted, normalized log-price. Define the transition densities $p_t(x, y) = \mathbb{P}(N_t < x, \langle N_t \rangle < y | t_0, N_{t_0}, \langle N \rangle_{t_0}) \mathbb{1}_{\{x \in \mathbb{R}, y \geq \langle N \rangle_{t_0}\}}$, and let $\hat{p}_t(z, w)$ be their characteristic functions. For all $(z, w) \in L_{k_1, k_2}$ we have:

$$\int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \left| e^{-iw\langle Y \rangle_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz} \Phi_{t_0}(-z, -w) \right| \hat{F}(z, w) dz dw \quad (8.11)$$

$$= \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \hat{p}_t(-z, -w) \hat{F}(z, w) dz dw = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{p}_t(-z + ik_1, -w + ik_2) \hat{F}(z + ik_1, w + ik_2) dz dw. \quad (8.12)$$

For $x \in \mathbb{R}, y \geq 0$, set $f(x, y) = e^{-k_1 x - k_2 y} F(x, y)$ and $g(x, y) = e^{k_1 x + k_2 y} p_t(x, y)$. We see that the integrand in (8.12) equals $\hat{f}(z, w) \hat{g}^*(z, w)$. But now f is $L^1(dx \times dy)$ because F is Fourier-integrable in Σ_F (for $(z, w) \in \Sigma_F$ take $\text{Re}(z) = \text{Re}(w) = 0$); similarly, \hat{g}^* is $L^1(dz \times dw)$ because of the L^1 assumption on Φ_{t_0} . Therefore, application of Parseval's formula (Titchmarsh, [41], Theorem 35) yields:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{p}_t(-z + ik_1, -w + ik_2) \hat{F}(z + ik_1, w + ik_2) dz dw = 4\pi^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_t(x, y) F(x, y) dx dy \quad (8.13)$$

$$= 4\pi^2 \mathbb{E}_{t_0}[F(N_t, \langle N \rangle_t)] < +\infty, \quad (8.14)$$

since $F \in L^1_{t_0}(N_t, \langle N \rangle_t)$. \square

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Notes

¹Jacod [23] uses the terminology T_t -adapted, and T_t -synchronized is sometimes found, but T_t -continuous is also common in the literature, and in our view less ambiguous.

²In general, time changes of Markov processes are not Markovian. By using Dambis, Dubins and Schwarz's Theorem ([27], Theorem 4.6) one can build a large class of counterexamples by starting from any continuous martingale which is not a Markov process.

³It is well known that in certain circumstances price densities associated through the canonical Feynman-Kac argument with some degenerate stochastic volatility models may be strict local martingales. This is a typical phenomenon occurring when the underlying diffusion problem does not admit a unique solution, as, for instance, in some GARCH-type models (compare with the counterexample provided). In the theory of time changes failure to meet the condition expressed in equation (3.8) is precisely the equivalent issue, and the case for "volatility explosions" ([32] Ch. 9) has just recreated in our context.

⁴Subordinated Brownian models are not covered, since these fail to satisfy the T_t -continuity assumption, which is worth recalling, we introduced as a *sufficient* condition for time-changed asset modeling. In particular, Theorem A does not apply to Brownian subordinated processes, thus in some sense valuation of volatility derivatives under such models represents an ill-posed problem.

⁵In [42], this author has independently found Φ_{t_0} in the Heston model by augmenting the SDE system associated to the problem (6.3)-(6.4) with the equation $dI_t = v_t dt$, and solved the associated Fourier-transformed parabolic equation via the usual Feynman-Kac argument. As it must be, the two approaches produce the same answer; therefore (5.2) also generalizes equation (3.8) of [42].

⁶Although Fang model has this assumption, our general framework notably allows for the presence of a correlation structure between W_t^1 , W_t^2 and W_t^3 .

⁷See [24], Ch. 1, 2.25, 2.26

Tables

Table 1: Parameter calibration from the S&P estimations of Fang [17], Sec. 4.

Parameters	Heston	Merton	Bates	Fang
σ_0	0.15	0.12	0.15	0.14
α	4.57		8.93	6.5
θ	0.0306		0.0167	0.0104
η	0.48		0.22	0.2
ρ	-0.82		-0.58	-0.48
λ_0		1.42	0.39	0.41
δ		0.0894	0.1049	0.2168
κ		-0.075	-0.11	-0.21
α_λ				5.06
θ_λ				0.13
η_λ				1.069

Table 2: $S_{t_0} = 100$, $K = 80$, $t_0 = 1$, $t = 2.5$, $r = 0.06$, $\bar{\sigma} = 0.1$, $TV_{t_0} = 0.23$.

	Heston	Merton	Bates	Fang
Analytical value	8.5801	8.5022	8.5575	8.8729
Monte Carlo	8.5660	8.5191	8.5511	8.8455

Table 3: $S_{t_0} = 100$, $K = 120$, $t_0 = 0.5$, $t = 4$, $r = 0.039$, $\bar{\sigma} = 0.1$, $TV_{t_0} = 0.08$.

	Heston	Merton	Bates	Fang
Analytical value	5.0899	5.7586	5.1129	5.0767
Monte Carlo	5.2121	5.8470	5.0908	5.0976

Table 4: $S_{t_0} = 100$, $K = 100$, $t_0 = 1.2$, $t = 1.5$, $r = 0.072$, $\bar{\sigma} = 0.1$, $TV_{t_0} = 0.31$.

	Heston	Merton	Bates	Fang
Analytical value	1.0155	1.1038	1.0258	1.0532
Monte Carlo	1.0119	1.0962	1.0259	1.0560

Table 5: $S_{t_0} = 100$, $K = 60$, $t_0 = 3$, $t = 6$, $r = 0.0225$, $\bar{\sigma} = 0.1$, $TV_{t_0} = 0.71$.

	Heston	Merton	Bates	Fang
Analytical value	12.2354	12.1637	12.3018	12.5606
Monte Carlo	12.2799	12.2105	12.3397	12.5284